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NEW UPPER BOUNDS FOR NONBINARY CODES

The paper presents new upper bounds for non-binary codes. The bounds can be obtained by linear and semidefinite programming.

bounds, codes, programming.

***Abstract* – New upper bounds on codes are presented. The bounds are obtained by linear and semidefinite programming.**

INTRODUCTION

One of the central problems in coding theory is to find upper bounds on maximum size $A_q(n, d)$ of a code of word length n and minimum Hamming distance at least d over the alphabet Q of $q \geq 2$ letters. Let us provide Q with the structure of an Abelian group, in an arbitrary way.

In 1973 Delsarte proposed a linear programming approach for bounding the size of cliques in an association scheme. This bound is based on diagonalizing the Bose-Mesner algebra of the scheme. To obtain bounds on $A_q(n, d)$, Delsarte introduced the Hamming scheme $H(n,q)$, which is generated by action of a group of permutations of Q^n that preserve the Hamming distance.

In 2005 Schrijver gave a new upper bound on $A_2(n,d)$ using semidefinite programming, which is obtained by block-diagonalizing the $(\frac{n+3}{3})$ – dimensional Terwilliger algebra of $H(n,2)$. The semidefinite programming bound for $A_q(n,d)$, based on the block-diagonalizing the $(\frac{n+4}{4})$ – dimensional Terwilliger algebra of $H(n, q)$ was presented later by Gijswijt, Schrijver and Tanaka.

In this work we introduce an association scheme which is generated by a subgroup of permutations of Q^n that preserve not only the Hamming distance, but also the "type" of the difference of vectors. The dimension of the Bose-Mesner algebra of this scheme is $(\frac{n+q-1}{q-1})$. We also describe the $(\frac{n+q^2-1}{q^2-1})$ – dimensional Terwilliger algebra of this new scheme. In particular, we have found that the orbits of Q^n x Q^n x Q^n under the action of the subgroup are characterized by certain q x q matrices.

With these two algebras in hand, we derive a linear programming bound and a semidefinite programming bound for $A_q(n, d)$ which generalize the bounds above. For the binary case, our scheme and the Hamming scheme $H(n, 2)$ coincide.

ASSOCIATION SCHEMES AND THE LP BOUND

Let G $-\{g_1 = 0, g_2, \ldots, g_{|G|}\}$ denote an (additively written) arbitrary finite abelian group with zero element 0, and $G^n = G \times G \times \ldots \times G$ denote an abelian group with respect to componentwise sum. For an integer n we denote

$$N^G_n := \{(\alpha_{g1}, \ldots, \alpha_{g|G|}) : \alpha_g \in \{0,1,\ldots,n\}, \sum_{g\in G} \alpha_g = n\}.$$

Define a function $\psi: G^n \to N^G_n$ as follows:

$$\psi\{x) := (c_{g1}\{x), c_{g2}\{x), \ldots, c_{g|G|}(x)),\ c_g(x) = |\{i : x_i = g\}|.$$

A nonempty subset C of G^n is called a code of length n. For a set $S \subseteq N_n^G$ we define

$A_G(n, S) := \max\{|C| : C \subseteq G^n, \psi\ (y - x) \in S\ \forall x, y \in C\}$.

For $\alpha \in N_n^G$ let R_α be a relation

$R_\alpha := \{(x, y) \in G^n \times G^n : \psi\ (y - x) = \alpha\}$

and denote $R = \{R_\alpha\}_{\alpha \in N^G_n}$.

Let H denote a group consisting of the permutations of G^n obtained by permuting the n coordinates followed by adding a word from G^n, i.e.,

$$H = \{\pi(\bullet) + \upsilon : \pi \in S_n, \upsilon \in G^n\}.$$

It is obvious that H acts transitively on G^n. H has a natural action on $G^n \times G^n$ given by $h(x,y) := (hx, hy)$. *The* following lemma states that the orbitals $\{(hx, hy) : h \in H\}$ form the relations of R.

Lemma 2.1. For any $a \in N^G_n$ and $x,y \in G^n$ such that $(x,y) \in R_\alpha$ there holds $R_\alpha = \{(hx,hy) : h \in H\}$.

Proof: Let $x, y \in X$ be such that $\psi\ (y - x) = \alpha$. Thus, for $h = \pi(\bullet) + v \in H$,

$hy - hx = (\pi y + \upsilon) - (\pi x + \upsilon) = \pi(y - x)$ and

$$\psi\ (hy - hx) = \psi\ (y - x) \tag{1}$$

which implies that $\{(hx, hy) : h \in H\} \subseteq R_\alpha$.

On the other hand, we have to show that for any $(\tilde{x}, \tilde{y}) \in R_\alpha$ there exists $h \in H$ such that $(\tilde{x}, \tilde{y}) = (hx, hy)$. One can see that exists $h_0 \in H$ such that $h_0x = 0$ and $h_0y - u_\alpha$, where

$$u_\alpha = \overbrace{0\ldots0}^{\alpha_0} \ldots \overbrace{g_i g_i \cdots g_i}^{\alpha_{gi}} \ldots \overbrace{g_{|G|} \cdots g_{|G|}}^{\alpha_{g|G|}},$$

namely $h_0(\bullet) = \pi_0(\bullet) - \pi_0 x$ for some $\pi_0 \in S_n$- Similarly, there exists $h_1 \in H$ such that $h_1\tilde{x} = 0$ and $h_1\tilde{y} - u_\alpha$, namely $h_1(\bullet) = \pi_1(\bullet) - \pi_1\tilde{x}$ for some $\pi_1 \in S_n$. Thus,

$$h(\bullet) = h_1^{-1}\ h_0(\bullet) = \pi_1^{-1}\ \pi_0(\bullet) + \tilde{x} - \pi_1^{-1}\ \pi_0(x)$$

satisfies $(hx, hy) = (\tilde{x}, \tilde{y})$ which proves the required inclusion.

Theorem 2.2. (G^n, R) is a commutative association scheme with $\binom{n+|G|-1}{|G|-1}$ relations.

Proof: It is well known (see for example [1]) that the orbitals from a group action form relations of an association scheme. For $(x, y) \in R_\gamma$ denote

$$Z_{(x,y)} := \{z \in G^n : (x,z) \in R_\alpha,\ (z,y) \in R_\beta\},$$
$$\hat{Z}_{(x,y)} = \{z \in G^n : (x,z) \in R_\beta,\ (z,y) \in R_\alpha\}.$$

Since $z \in Z_{(x,y)} \leftrightarrow (-z) \in \hat{Z}_{(-y,-x)}$ we conclude that

$$p^\gamma_{\alpha,\beta} = |Z(x,y)| = |\hat{Z}_{(-y,-x)}| = p^\gamma_{\beta,\alpha}.$$

Note that the number of relations is equal to the number of $(|G| - 1)$ – tuples of nonnegative integers $(\alpha_{g2}, \ldots, \alpha_{g|G|})$ such that $\alpha_{g2} + \ldots + \alpha_{g|G|} \leq n$.

Let D_α denote the adjacency matrix of the relation R_α, i.e.,

$$(D_\alpha)_{x,y} = \begin{cases} 1, if\ (x,y) \in R_\alpha, \\ 0, otherwise. \end{cases}$$

The matrices $\{D_\alpha\}_{\alpha} \in N_n^G$ form a basis of a commutative $\binom{n+|G|-1}{|G|-1}$ – dimensional Bose-Mesner algebra A_{Gn} of the scheme (G^n, R).

In general, (G^n, R) is a non-symmetric association scheme.

For $\alpha \in N^G_n$, the inverse $R_\alpha^{-1} = \{(y, x) : (x, y) \in R_\alpha\}$ of the relation R_α is given by $R_\alpha^{-1} = R_{\hat{\alpha}}$ where

$$\hat{\alpha} := (\hat{\alpha}_{g2}, \ldots, \hat{\alpha}_{g|G|}),\ \hat{\alpha}_{gi} = \alpha_{-g_i}. \tag{2}$$

It's easy to see that the valency of the relation R_α (and of $R_{\hat{\alpha}}$) is $\upsilon_\alpha = p_{\alpha\hat{\alpha}}^{(n,0,\ldots,0)} = \binom{n}{\alpha_0, \alpha_{g2}, \ldots, \alpha_{g|G|}}$.

Consider the association scheme $(G^n, \tilde{R})$, where $\tilde{R} = \{\tilde{R}_\alpha\}$, $\tilde{R}_\alpha = R_\alpha \cup R_\alpha^{-1}$. This is symmetric association scheme. Note that

$$\widetilde{D_\alpha} = D_\alpha + D_{\hat{\alpha}}$$

are symmetric matrices. We denote by $\tilde{A}_{Gn}$ the Bose-Mesner algebra of $(G^n, \tilde{R})$ and by $\widehat{G^n} = \{X_u\}_{u \in G^n}$ the group of characters. The next theorem gives more details about the symmetric scheme.

Theorem 2.3. The unitary matrix U which diagonalizes the $\tilde{A}_{Gn}$ is given by

$$(U)_{x,y} = \frac{1}{|G|^{n|2}} X_x(y)\ .$$

The primitive idempotent $\tilde{J}_\alpha$, $\alpha \in N_n^G$, is the matrix with (x, y) entry

$$(\tilde{J}_\alpha)_{x,y} = \frac{1}{|G|^n} \sum_{\substack{z\in G^n \\ \varphi(z)\epsilon\{\alpha,\hat{\alpha}\}}} X_{y-x}(z)\ . \quad (3)$$

The eigenvalues are given by

$$\tilde{P}_\beta(\alpha) = Q_\beta(\alpha) = \sum_{\substack{z\in G^n \\ \varphi(z)\epsilon\{\alpha,\hat{\alpha}\}}} X_u(z) \quad (4)$$

where $u \in G^n$ is any word with $\psi(u) \in \{\alpha,\hat{\alpha}\}$.

For $\alpha = (\alpha_0, \ldots, \alpha_{g|G|}) \in N_n^G$ there holds

$$K_k\left(\sum_{g\in G*} \alpha_g\right) = \sum_{\substack{\beta=(\beta_0,\ldots,\beta_{g|G|})\in N_n^G \\ \beta_{g2}+\cdots+\beta_{g|G|=k}}} \tilde{Q}_\beta(\alpha).$$

Where K_k(x) is the Krawtchouk polynomial of degree k.

A. Association Scheme for $G = Z_3$.
Let us look at an example for $G = Z_3 = \{0, 1, 2\}$. For convenience we will omit a_0.

$$N_n^G = \{\alpha - (\alpha_1, \alpha_2) : \alpha_1 + \alpha_2 \le n\}.$$

Thus, the number of relations in a non-symmetric scheme (Z_3^n, R) is |R| = $= \binom{n+2}{2}$, and the number of relations in the symmetric scheme $(Z_3^n, \tilde{R})$ is

$$|\tilde{R}| = \begin{cases} \frac{(n+2)^2}{4}, & if\ n\ is\ even, \\ \frac{(n+1)(n+3)}{4}, & if\ n\ is\ odd. \end{cases}$$

The polynomial $\tilde{Q}_{(\beta_1,\beta_2)}((\alpha_1,\alpha_2)) = \sum_{\substack{p+q\le\alpha_1 \\ s+t\le\alpha_2 \\ p+s\le\beta_1 \\ q+t\le\beta_2}} \binom{n-\alpha_1-\alpha_2}{\beta_1-p-s,\beta_2-q-t}\binom{\alpha_1}{p,q}\binom{\alpha_2}{s,t} \times$
$\times\, e^{2\pi i/3(p+q+s+t)}\left(e^{2\pi i/3(q+s)} + (1-\delta_{\beta_1,\beta_2})e^{2\pi i/3(p+t)}\right).$
We list here few polynomials:

$$\tilde{Q}_{(0,0)}((\alpha_1,\alpha_2)) \equiv 1,$$

$$\tilde{Q}_{(1,0)}\big((\alpha_1,\alpha_2)\big) = 2n - 3(\alpha_1+\alpha_2),$$

$$\tilde{Q}_{(1,1)}\big((\alpha_1,\alpha_2)\big) = 2\binom{n-\alpha_1-\alpha_2}{2} -$$

$$-(\alpha_1+\alpha_2)\big(n-(\alpha_1+\alpha_2)\big) - \alpha_1\alpha_2 + 2\binom{\alpha_1}{2} + 2\binom{\alpha_2}{2},$$

$$\tilde{Q}_{(2,0)}\big((\alpha_1,\alpha_2)\big) = 2\binom{n-\alpha_1-\alpha_2}{2} -$$

$$-(\alpha_1+\alpha_2)\big(n-(\alpha_1+\alpha_2)\big) + 2\alpha_1\alpha_2 - \binom{\alpha_1}{2} - \binom{\alpha_2}{2},$$

$$\tilde{Q}_{(n,0)}\big((\alpha_1,\alpha_2)\big) = \begin{cases} 2 \ if \ \alpha_1 \equiv \alpha_2 (mod\ 3), \\ -1\ else. \end{cases}$$

B. The Linear Programming Bound.

For a code $C \in G^n$ let $(a_\gamma)_\gamma \in N_n^G$ denote the inner distribution of C, i.e.,

$$a_\gamma = \frac{|\tilde{R}_\gamma \cap C \times C|}{|C|}.$$

Clearly, we have

$$a_{(n,0,\ldots,0)} = 1,\ \sum_{\gamma \in N_n^G} a_\gamma = |C|.$$

The Delsarte's linear programming bound is given in the following theorem.

Theorem 2.4. (LP bound) For any positive integer n and set $S \subseteq N_n^G$ such that $(n, 0, ..., 0) \in S$

$$A_G(n,S) \le \left\lfloor max \sum_{\gamma \in N_n^G} a_\gamma \right\rfloor$$

subject to the constraints

$a_{(n,0,\ldots,0)} = 1$,

$a_\gamma = 0$ for $\gamma \notin S$,

$$\sum_{\gamma \in N_n^G} \tilde{Q}_\alpha(\gamma) a_\gamma \ge 0, \alpha \in N_n^G.$$

Where $\tilde{Q}_\alpha(\gamma)$ is given in (4).

THE TERWILLIGER ALGEBRA OF (G^n, R)

We will now consider the action of H on, ordered triples of words, leading to noncommutative algebra $\mathcal{T}_{G^n}$ containing the Bose-Mesner algebra. Let $M,\{G)$ be the following set of matrices:

$M_n(G) := A \epsilon \mathbb{C}^{|G| \times |G|}$: $(A)_{g_i, g_j} \in \{0,1,\dots,n\}$ and $\sum_{g_i, g_j} (A)_{g_i, g_j} = n\}$.

For any matrix $A \in M_n(G)$ we define three vectors $r(A)$, $c(A)$, $p(A) \in \mathrm{N}_n^G$ by

$$r(A) = \left(\sum_{g \in G} (A)_{0,g}, \dots, \sum_{g \in G} (A)_{g|G|,g}\right),$$

$$c(A) = \left(\sum_{g \in G} (A)_{0,g}, \dots, \sum_{g \in G} (A)_{g,g|G|}\right),$$

$$p(A) = \left(\sum_{g \in G} (A)_{g,g}, \sum_{g \in G} (A)_{g(g+g2)}, \dots, \sum_{g \in G} (A)_{g,(g+g|G|)}\right). \tag{5}$$

To each ordered triple (x, y, z) $\in G^n$ xG^n x G^n we associate the matrix

$$\hat{\psi}(x,y,z) := A_{y,z}^x \in M_n(G)$$

where

$$(A_{y,z}^x)_{g_i, g_j} := \left|\{k: (y-x)k = g_i, (z-x)k = g_j\}\right|.$$

Note that $\psi(y-x)$, $\psi(z-x)$ and $\psi(z-y)$ are uniquely determined by the $A_{y,z}^x$:

$$\psi(y-x) = r(A_{y,z}^x),\ \psi(z-x) = c\,(A_{y,z}^x),\ \psi(z-y) = p\,(A_{y,z}^x). \tag{6}$$

If we define

$X_A := \{x, y, z\} \in G^n \times G^n : \hat{\psi}(x, y, z) = A\}$

for $A \in M_n(G)$, we have the following.

Lemma 3.1. The sets X_A, $A \in M_n(G)$, are the orbits of G^n x G^n x G^n under the action of H.

Proof: Let x, y, z $\in G^n$ and let $\hat{\psi}$ (x, y, z) = A. For h = $\pi(\bullet) + v \in H$ we have from (1)

$$(\hat{\psi}(hx, hy, hz))_{gi,gj} = |\{k: (hy-hx)_k = g_i, (hz-hx)_k = g_j\}| = |\{k: (\pi(y-x))_k = g_i,$$

$$(\pi(z-x))_k = g_j\}| = |\{k: (y-x)_k = g_i, (z-x)_k = g_j\}| = (A)_{gi,gj} = (\hat{\psi}(x,y,z))_{gi,gj},$$

which implies

$$\hat{\psi}(x, y, z) = \hat{\psi}(hx, hy, hz)$$

for any h∈H.

Let A ∈M_n(G). To show that H acts transitively on X_A it suffices to show that for every *(x, y, z)* ∈X_A there is h∈H such that *(hx, hy, hz)* only depends on *A*. For convenience, we denote ψ(y – x) = α and ψ *(z – x)* = β. Let π_0∈S_n be such that

$$\pi_0(y-x)=u_\alpha=\overbrace{0\ldots\ldots0}^{\alpha_0}\ldots\overbrace{g_ig_i\cdots g_i}^{\alpha_{gi}}\ldots\overbrace{g_{|G|}\cdots g_{|G|}}^{\alpha_{g|G|}}.$$

Now, let $\pi 1$∈S_n be such that

$$\pi_1 u_\alpha \text{ and } \pi_1\pi_0(z-x) = u_\beta,$$

where

α	$\overbrace{0\ldots.0}^{\alpha_0}$	..	. $\overbrace{g_ig_i\cdots g_i}^{\alpha_{g_i}}$		$\overbrace{g_{\|G\|}\cdots g_{\|G\|}}^{\alpha_{g_{\|G\|}}}$
β	…	…	*0…0(A)gi,0…g|G|…g|G|(A)gi,g|G|*		…

Thus,

$$h = \pi_1\pi_0(\bullet) - \pi_1\pi_0(x).$$

Denote the stabilizer of 0 ∈ G^n in *H* by H_0. For A ∈M_n (G), let M_A be the $|G|^n \times |G|^n$ matrix defined by:

$$(M_A)_{y,z} := \begin{cases} 1, if\ \hat{\psi}(0, y, z) = A, \\ 0,\ otherwise. \end{cases}$$

Note that

$$M_A^T = M_{A^T}.$$

Let τ_{G^n} be the set of matrices

$$\sum_{A\in M_{n(G)}} x_A M_A,$$

where x_A ∈ ℂ. From the Lemma 3.1 it follows that τ_{G^n} is the set of matrices that are stable under permutations σ∈H_0 of the rows and columns, i.e., fo any σ∈H_0 and M_A,

$$(M_A)_{y,z} = (M_A)_{\sigma y, \sigma z}\,.$$

Hence τ_{G^n} is a complex matrix algebra called the *centralizer algebra* of H_0. Since

$$M_A M_B = 0 \text{ if } c(A) \neq r\{B)\,.$$

it follows that τ_{G^n} is a noncommutative algebra. The M_A constitute a basis for τ_{G^n}, and hence

$$\dim \tau_{G^n} = |M_n(G)| = \binom{n+|G|^2-1}{|G|^2-1}.$$

Note that the algebra τ_{G^n} contains the Bose-Mesner algebra A_{G^n}; for $\gamma \in N_n^G$ we have (recall (5))

$$D_\gamma = \sum_{\substack{A \in M_{n(G)} \\ p(A)=\gamma}} M_A,$$

Let τ denote the Terwilliger algebra of the association scheme (G^n, R) (with respect to 0). It is the complex matrix algebra generated by the adjacency matrices of the scheme $\{D_\gamma\}_\gamma \in N_n^G$ and the diagonal matrices $\{E_\gamma^*\}_\gamma \in N_n^G$ defined by

$$(E_\gamma^*)_{x,x} := \begin{cases} 1 & if\ (0,x) \in R_\gamma \\ 0 & othewise \end{cases}$$

Theorem 3.2. The algebras τ_{G^n} and τ coincide.

Proof: We have already seen in (7) that τ_{G^n} contains the adjacency matrices D_γ. Note that

$$E_\gamma^* = M_{\Delta\gamma},$$

where $\Delta_\gamma = \text{diag}(\gamma_0, \gamma_{g2}, \ldots, \gamma_{g|G|}) \in M_n\{G)$. Hence τ is a subalgebra of τ_{G^n}. Now we show the reverse inclusion. For $\gamma \in N_n^G$ with $\gamma_{gi} \geq k$ and $g_j \in G$, $g_i \neq g_j$, define

$$\gamma(k, g_i, g_j) \in N_n^G \text{ by}$$

$$(\gamma(k,g_i,g_j))_{gl} := \begin{cases} \gamma_{gl} & if\ l \neq i,j, \\ \gamma_{g_i} - k & if\ l = i, \\ \gamma_{g_j} + k & if\ l = j. \end{cases}$$

Also define the zero-one matrices:

$$N_{\gamma}(k,g_i,g_j) = E_{\gamma}^{*}D_{(n\text{-}k,\ 0,\ \dots,\ 0,k,0,\dots,0)}\ E_{\gamma(k,g_i,g_j)}^{*}$$

where at the index of the matrix D, k appears in the $(g_j - g_i)$ coordinate. Observe that

$$(N_{\gamma}(k,g_i,g_j))_{y,z} = 1 \leftrightarrow (0, y, z) \in X_A,$$

Where

$$(A)_{gl,\ gm} = \begin{cases} \gamma_{gl}\ if\ l = m\ and\ l \neq i, \\ \gamma_{g_i - k}\ \ if\ l = m = i, \\ k\ \ if\ (l,m) = (i,j), \\ 0\quad otherwise. \end{cases}$$

I. Semidefinite Programming Bound

For $h \in H$ denote the characteristic vector of $h(C)$ by $X^{(hC)}$ (taken as a column vector). For a word $x \in G^n$, let $h_x \in H$ be any automorphism with $h_x(x) = 0$, and define

$$R_x = \frac{1}{|H_0|}\sum_{\sigma\epsilon H_0} X^{\sigma(h_x(C))}(X^{\sigma(h_x(C))})^T.$$

Next define the matrices R and R' by

$$R := \frac{1}{|C|}\ \Sigma_{x\in C} R_x,$$
$$R' := \frac{1}{(|G|^n - |C|}\ \Sigma_{x\in G^n\setminus C} R_x.$$

As the R_x, and hence also R and R', are convex combinations of positive semidefinite matrices, they are positive semidefinite. By construction, the matrices R_x, and hence the matrices R and R' are invariant under permutations $\sigma \in H_0$ of rows and columns and hence they are elements of the algebra To^*. Define the numbers

$$\lambda_A := |\ (C\ xC\ xC)\cap X_A|$$

and let

$$\mu a := |(\{0\}xG"xG")\cap X_A|$$

be the number of nonzero entries of Ma. It is easy to see that

$$\mu_A = \binom{n}{r(A)_0, r(A)_{g2}, \dots, r(A)_{g|G|}} \times \prod_{g_i \in G}\binom{r(A)_{g_i}}{(A)_{g_i,0,}(A)_{g_i,g_2}, \dots, (A)_{g_i,g_{|G|}}}$$

Theorem 4.1.

$$R=\sum_{A\in M_n(G)} x_A M_A,$$
$$R'=\frac{|C|}{|C|^n-|C|}\sum_{A\in M_n(G)}\left(x\Delta_{p(A)}-x_A\right)M_A,$$

where

$$x_A=(|C|_{\mu_A})^{-1}\lambda_A.$$

Proof: Denote by *(A,B) := tr(A*B),* the standard inner product on the space of complex $|G^n|$ x |G"| matrices. Observe that the matrices M_A are pairwise orthogonal and that $(M_A,M_A)=\mu_A$ for A G M_n *(G).* Hence

$$\langle R,M_A\rangle=\frac{1}{|C|}\sum_{x\in C}\langle R_x,M_A\rangle=\frac{1}{|C|}\sum_{x\in C}|(\{x\}\times C\times C)\cap X_A|=\frac{1}{|C|}\lambda_A$$

implies that

$$\sum_{(0,y,z)\in X_A}(R)_{y,z}=\lambda_A$$

which is the total number of l's (with repetitions) in positions where R and M_A are both nonzero. From the symmetry, each entry in M_A is counted the same number of times which is $(\mu_A)^{-1}\lambda_A$. Thus the first claim follows:

$$R=\sum_{A\in M_n(G)}\frac{1}{\mu_A}\langle R,M_A\rangle=\sum_{A\in M_n(G)}x_AM_A.$$

Now, the matrix

$$T:=|C|R+(|G|^n-|C|R'=\sum_{x\in G^n}R_x=$$
$$=\frac{1}{|H_0|}\sum_{x\in G^n}\sum_{\sigma\in H_0}x^{\sigma(h_x(C))}(x^{\sigma(h_x(C))})^T=$$
$$=\frac{1}{|H_0|}\sum_{x\in G^n}\sum_{\sigma\in H_0}x^{\sigma(C)}(x^{\sigma(C)})^T=\frac{1}{|H_0|}\sum_{h\in H}x^{h(C)}(x^{h(C)})^T$$

is invariant under permutation of the rows and columns by permutations h ∈ H and hence is an element of the Bose-Mesner algebra say

$$T = \sum_{\gamma \in N_n^G} b_\gamma D_\gamma.$$

Note that for any z ∈ G^n with ψ(z) = γ, we have

$$b_\gamma = (T)_{0,z} = |C|(R)_{0,z} + (|G|^n - |C|)(R')_{0,z}.$$

From the definition of R'

$$R' := \frac{1}{(|G|^n - |C|)} \sum_{x \in G^n \setminus C} R_x$$

follows that for x ∈ G^n \ C, holds $0 \neq h_x(C)$ and $0 \neq \sigma(h_x(C))$ for any σ ∈ H_0. Therefore, $(R')_{0,z} = 0$ and we obtain

$$b_\gamma = (T)_{0,z} = |C|(R)_{0,z} = |C| \sum_{A \in M_n(G)} x_A (M_A)_{0,z} = |C| x_{(\gamma)},$$

where (γ) denotes a matrix whose first row is a vector γ and the rest of the rows are zero vectors. Hence we have

$$(|G|^n - |C|)R' = T - |C|\, R = |C| \sum_{A \in M_n(R)} (x_{(p(A))} - x_A) M_A.$$

Finally, note that

$$x_{(p(A))} = (|C|_{\mu_{(p(A))}})^{-1} \lambda_{(p(A))} = \frac{1}{|C|_{v_{p(A)}}} |C \times C \cap R_{p(A)}| = \frac{a_{p(A)}}{v_{p(A)}} =$$
$$= (|C|_{\mu\Delta_{p(A)}} = x\Delta_{p(A)}.$$

Since for $\gamma \in N_n^G$,

$$\lambda_{\Delta_\gamma} := \left|(C \times C \times C) \cap X_{\Delta_\gamma}\right| = \sum_{c \in C} \sum_{\substack{c' \in C \\ \psi(c'-c)=\gamma}} 1.$$

It follows that

$$\sum_{\gamma \in N_n^G} \lambda_{\Delta_\gamma} = \sum_{\gamma \in N_n^G} \sum_{c \in C} \sum_{\substack{c' \in C \\ \psi(c'-c)=\gamma}} 1 = \sum_{c \in C} \sum_{\gamma \in N_n^G} \sum_{\substack{c' \in C \\ \psi(c'-c)=\gamma}} 1 = \sum_{c \in C} \sum_{c' \in C} 1 = |C|^2$$

and hence we have

$$|C|^2 = \sum_{\gamma\in N_n^G} \lambda_{\Delta_\gamma} = \sum_{\gamma\in N_n^G} |C|_{\mu_{\Delta_\gamma} x_{\Delta_\gamma}}.$$

Therefore

$$|C| = \sum_{\gamma\in N_n^G} \mu_{\Delta_\gamma} x_{\Delta_\gamma} = \sum_{\gamma\in N_n^G} \binom{n}{\gamma_0, \gamma_{g2}, \dots, \gamma_{g|G|}} x_{\Delta_\gamma} = \sum_{\gamma\in N_n^G} \upsilon_\gamma x_{\Delta_\gamma}.$$

Now we are ready to formulate the bound.

Theorem* 4.*2. (SDP bound) For any positive integer n **and** set $S \subseteq N_n^G$ such that $(n, 0,\dots, 0) \in S$

$$A_G(n, S) \le \left\lfloor \max \sum_{\gamma\in N_n^G} \upsilon_\gamma x_{\Delta_\gamma} \right\rfloor$$

subject to the constraints

$$x_{\Delta_{(n,0,\dots,0)}} = 1,$$

$$x_{A = 0} \; if \; \{r(A), \quad c(A), \quad p(A)\} \nsubseteq S,$$

$$x_A = x_{A^T},$$

$$R \succcurlyeq 0, R' \succcurlyeq 0.$$

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Б. Маунитс

НОВЫЕ ВЕРХНИЕ ГРАНИЦЫ НЕБИНАРНОГО КОДА

Статья посвящена изучению верхних границ небинарных кодов. Границы можно получить путем линейного или полуопределенного программирования.

границы, коды, программирование.