NEW UPPER BOUNDS FOR NONBINARY CODES

The paper presents new upper bounds for non-binary codes. The bounds can be obtained by linear and semidefinite programming.

Abstract – New upper bounds on codes are presented. The bounds are obtained by linear and semidefinite programming.

INTRODUCTION

One of the central problems in coding theory is to find upper bounds on maximum size $A_q(n, d)$ of a code of word length $n$ and minimum Hamming distance at least $d$ over the alphabet $Q$ of $q \geq 2$ letters. Let us provide $Q$ with the structure of an Abelian group, in an arbitrary way.

In 1973 Delsarte proposed a linear programming approach for bounding the size of cliques in an association scheme. This bound is based on diagonalizing the Bose-Mesner algebra of the scheme. To obtain bounds on $A_q(n, d)$, Delsarte introduced the Hamming scheme $H(n, q)$, which is generated by action of a group of permutations of $Q$ that preserve the Hamming distance.

In 2005 Schrijver gave a new upper bound on $A_2(n, d)$ using semidefinite programming, which is obtained by block-diagonalizing the $\left(\begin{array}{c}n+3 \cr 3 \end{array}\right)$-dimensional Terwilliger algebra of $H(n, 2)$. The semidefinite programming bound for $A_q(n, d)$, based on the block-diagonalizing the $\left(\begin{array}{c}n+4 \cr 4 \end{array}\right)$-dimensional Terwilliger algebra of $H(n, q)$ was presented later by Gijswijt, Schrijver and Tanaka.

In this work we introduce an association scheme which is generated by a subgroup of permutations of $Q^q$ that preserve not only the Hamming distance, but also the "type" of the difference of vectors. The dimension of the Bose-Mesner algebra of this scheme is $\left(\begin{array}{c}n+q-1 \cr q-1 \end{array}\right)$. We also describe the $\left(\begin{array}{c}n+q^2-1 \cr q^2-1 \end{array}\right)$-dimensional Terwilliger algebra of this new scheme. In particular, we have found that the orbits of $Q^q \times Q^q \times Q^q$ under the action of the subgroup are characterized by certain $q \times q$ matrices.

With these two algebras in hand, we derive a linear programming bound and a semidefinite programming bound for $A_q(n, d)$ which generalize the bounds above. For the binary case, our scheme and the Hamming scheme $H(n, 2)$ coincide.

ASSOCIATION SCHEMES AND THE LP BOUND
Let $G = \{g_1 = 0, g_2, \ldots, g_n\}$ denote an (additively written) arbitrary finite abelian group with zero element 0, and $G^n = G \times G \times \ldots \times G$ denote an abelian group with respect to componentwise sum. For an integer $n$ we denote

$$N^n_n := \{(a_{g1}, \ldots, a_{g|G|}) : a_g \in \{0,1, \ldots, n\}, \sum_{g \in G} a_g = n\}.$$ 

Define a function $\psi : G^n \rightarrow N^n_n$ as follows:

$$\psi(x) := (c_{g1}(x), c_{g2}(x), \ldots, c_{g|G|}(x)), \ c_g(x) = |\{i : x_i = g\}|.$$ 

A nonempty subset $C$ of $G^n$ is called a code of length $n$. For a set $S \subseteq N^n_n$ we define

$$A_G(n, S) := \max \{|C| : C \subseteq G^n, \ \psi(y-x) \in S \ \forall x, y \in C\}.$$ 

For $a \in N^n_n$ let $R_a$ be a relation

$$R_a := \{(x, y) \in G^n \times G^n : \psi(y-x) = a\}$$

and denote $R = \{R_a \} \ \ a \in G^n$.

Let $H$ denote a group consisting of the permutations of $G^n$ obtained by permuting the $n$ coordinates followed by adding a word from $G^n$, i.e.,

$$H = \{\pi(\bullet) + u : \pi \in S_n, u \in G^n\}.$$ 

It is obvious that $H$ acts transitively on $G^n$. $H$ has a natural action on $G^n \times G^n$ given by $h(x, y) := (hx, hy)$. The following lemma states that the orbitals $\{(hx, hy) : h \in H\}$ form the relations of $R$.

**Lemma 2.1.** For any $a \in N^n_n$, and $x, y \in G^n$ such that $(x, y) \in R_a$, there holds $R_a = \{(hx, hy) : h \in H\}$.

**Proof:** Let $x, y \in X$ be such that $\psi(y-x) = a$. Thus, for $h = \pi(\bullet) + u \in H$,

$$hy - hx = (\pi y + u) - (\pi x + u) = \pi(y-x)$$

and

$$\psi(hy-hx) = \psi(y-x)$$

which implies that $\{(hx, hy) : h \in H\} \subseteq R_a$.

On the other hand, we have to show that for any $(\bar{x}, \bar{y}) \in R_a$ there exists $h \in H$ such that $(\bar{x}, \bar{y}) = (hx, hy)$. One can see that exists $h_0 \in H$ such that $h_0x = 0$ and $h_0y - u_{a_0}$ where $a_0 = a_{g_1} a_{g_2} \ldots a_{g|G|}$

$$u_a = g_1 \ldots g_l \ldots g_{|G|},$$

namely $h_0(\bullet) = \pi_0(\bullet) - \pi_0x$ for some $\pi_0 \in S_n$. Similarly, there exists $h_1 \in H$ such that $h_1 \bar{x} = 0$ and $h_1 \bar{y} - u_{a_1}$, namely $h_1(\bullet) = \pi_1(\bullet) - \pi_1 \bar{x}$ for some $\pi_1 \in S_n$. Thus,

$$h(\bullet) = h_{1}^{-1} h_0(\bullet) = \pi_{1}^{-1} \pi_0(\bullet) + \bar{x} - \pi_{1}^{-1} \pi_0(x)$$
satisfies \((hx, hy) = (\bar{x}, \bar{y})\) which proves the required inclusion.

Theorem 2.2. \((G^n, R)\) is a commutative association scheme with \(\binom{n+|G|-1}{|G|-1}\) relations.

Proof: It is well known (see for example [1]) that the orbitals from a group action form relations of an association scheme. For \((x, y) \in R\), denote

\[
Z_{(x,y)} = \{z \in G^n : (x, z) \in R_{\alpha}, (z, y) \in R_{\beta}\},
\]

\[
\tilde{Z}_{(x,y)} = \{z \in G^n : (x, z) \in R_{\beta}, (z, y) \in R_{\alpha}\}.
\]

Since \(z \in Z_{(x,y)} \leftrightarrow (z) \in \tilde{Z}_{(x,y)}\) we conclude that

\[
p^\gamma_{\alpha, \beta} = |Z(x, y)| = |\tilde{Z}_{(x,y)}| = p^\gamma_{\beta, \alpha}.
\]

Note that the number of relations is equal to the number of \(|G| - 1\) - tuples of nonnegative integers \((\alpha_{g2}, \ldots, \alpha_{g|G|})\) such that \(\alpha_{g2} + \ldots + \alpha_{g|G|} \leq n\).

Let \(D_{a}\) denote the adjacency matrix of the relation \(R_{a}\), i.e.,

\[
(D_{a})_{x,y} = \begin{cases} 1, & \text{if } (x, y) \in R_{a}, \\ 0, & \text{otherwise}. \end{cases}
\]

The matrices \(\{D_{a}\}_{a} \in N^n_G\) form a basis of a commutative \(\binom{n+|G|-1}{|G|-1}\)-dimensional Bose-Mesner algebra \(A_{G^n}\) of the scheme \((G^n, R)\).

In general, \((G^n, R)\) is a non-symmetric association scheme. For \(\alpha \in N^n_G\), the inverse \(R_{a}^{-1} = \{(y, x) : (x, y) \in R_{a}\}\) of the relation \(R_{a}\) is given by \(R_a^{-1} = R_{\bar{a}}\) where

\[
\bar{\alpha} := (\bar{\alpha}_{g2}, \ldots, \bar{\alpha}_{g|G|}), \quad \bar{\alpha}_{gi} = \alpha_{-gi}.
\]

\(\bar{\alpha}\)'s easy to see that the valency of the relation \(R_{a}\) (and of \(R_{\bar{a}}\)) is \(u_a = p^a_{(n,0,\ldots,0)} = (a_{g2}, \ldots, a_{g|G|})\).

Consider the association scheme \((G^n, \bar{R})\), where \(\bar{R} = \{\bar{R}_{a}\}, \bar{R}_{a} = R_{a} \cup R_{a}^{-1}\). This is symmetric association scheme. Note that

\[
\bar{D}_{a} = D_{a} + D_{\bar{a}}
\]

are symmetric matrices. We denote by \(\bar{A}_{G^n}\) the Bose-Mesner algebra of \((G^n, \bar{R})\) and by \(\bar{G}^n = \{X_u\}_{u \in G^n}\) the group of characters. The next theorem gives more details about the symmetric scheme.

Theorem 2.3. The unitary matrix \(U\) which diagonalizes the \(\bar{A}_{G^n}\) is given by
The primitive idempotent $T_\alpha$, $\alpha \in N_\mathbb{G}$, is the matrix with $(x, y)$ entry

$$(f_\alpha)_{x,y} = \frac{1}{|G|^2} \sum_{z \in G} X_{y-x}(z).$$

The eigenvalues are given by

$$\tilde{p}_\beta(\alpha) = Q_\beta(\alpha) = \sum_{z \in G} X_u(z)$$

where $u \in G^n$ is any word with $\psi(u) \in \{\alpha, \hat{\alpha}\}$.

For $\alpha = (\alpha_0, \ldots, \alpha_{g|G|}) \in N_\mathbb{G}$ there holds

$$K_k \left( \sum_{g \in G^*} \alpha_g \right) = \sum_{\beta \in G^*} \tilde{Q}_\beta(\alpha).$$

Where $K_k(x)$ is the Krawtchouk polynomial of degree $k$.

### A. Association Scheme for $G = Z_3$

Let us look at an example for $G = Z_3 = \{0, 1, 2\}$. For convenience we will omit $a_0$.

$N_\mathbb{G} = \{\alpha - (\alpha_1, \alpha_2) : \alpha_1 + \alpha_2 \leq n\}$.

Thus, the number of relations in a non-symmetric scheme $(Z_3^n, R)$ is $|R| = \binom{n+2}{2}$, and the number of relations in the symmetric scheme $(Z_3^n, \bar{R})$ is

$$|\bar{R}| = \begin{cases} \frac{(n+2)^2}{4}, & \text{if } n \text{ is even,} \\ \frac{(n+1)(n+3)}{4}, & \text{if } n \text{ is odd.} \end{cases}$$

The polynomial $\bar{Q}_{(0,0)}((\alpha_1, \alpha_2)) = \sum_{p+q+s+t \leq n} \binom{n-\alpha_1-\alpha_2}{\beta_1-p-s, \beta_2-q-t} \binom{\alpha_1}{p, q, s, t} \times e^{2\pi i/3(p+q+s+t)}(e^{2\pi i/3(q+s)} + (1 - \delta_{\beta_1, \beta_2})e^{2\pi i/3(p+t)}).

We list here few polynomials:

$$\bar{Q}_{(0,0)}((\alpha_1, \alpha_2)) \equiv 1,$$
\[ \tilde{Q}_{(1,0)}((\alpha_1, \alpha_2)) = 2n - 3(\alpha_1 + \alpha_2), \]
\[ \tilde{Q}_{(1,1)}((\alpha_1, \alpha_2)) = 2^{n-\frac{\alpha_1 - \alpha_2}{2}} - (\alpha_1 + \alpha_2)(n - (\alpha_1 + \alpha_2)) - \alpha_1\alpha_2 + 2 \left(\frac{\alpha_1}{2}\right) + 2 \left(\frac{\alpha_2}{2}\right), \]
\[ \tilde{Q}_{(2,0)}((\alpha_1, \alpha_2)) = 2^{n-\frac{\alpha_1 - \alpha_2}{2}} - (\alpha_1 + \alpha_2)(n - (\alpha_1 + \alpha_2)) + 2\alpha_1\alpha_2 - \left(\frac{\alpha_1}{2}\right) - \left(\frac{\alpha_2}{2}\right), \]
\[ \tilde{Q}_{(n,0)}((\alpha_1, \alpha_2)) = \begin{cases} 2 & \text{if } \alpha_1 \equiv \alpha_2 \pmod{3}, \\ -1 & \text{else}. \end{cases} \]

**B. The Linear Programming Bound.**

For a code \( C \in G^n \) let \( (a_\gamma)_{\gamma} \in N_n^G \) denote the inner distribution of \( C \), i.e.,

\[ a_\gamma = \frac{|\tilde{K}_\gamma \cap C \times C|}{|C|}. \]

Clearly, we have

\[ a_{(n,0,...,0)} = 1, \sum_{\gamma \in N_n^G} a_\gamma = |C|. \]

The Delsarte's linear programming bound is given in the following theorem.

**Theorem 2.4.** (LP bound) For any positive integer \( n \) and set \( S \subseteq N_n^G \) such that \((n, 0, ..., 0) \in S\)

\[ A_G(n, S) \leq \max \sum_{\gamma \in N_n^G} a_\gamma \]

subject to the constraints

\[ a_{(n,0,...,0)} = 1, \]
\[ a_\gamma = 0 \text{ for } \gamma \notin S, \]
\[ \sum_{\gamma \in N_n^G} \tilde{Q}_\alpha(\gamma) a_\gamma \geq 0, \alpha \in N_n^G. \]

Where \( \tilde{Q}_\alpha(\gamma) \) is given in (4).
THE TERWILLIGER ALGEBRA OF \((G^n, R)\)

We will now consider the action of \(H\) on, ordered triples of words, leading to noncommutative algebra \(T_{G^n}\) containing the Bose-Mesner algebra. Let \(M_n(G)\) be the following set of matrices:

\[
M_n(G) := \{ A \in \mathbb{C}^{G^n \times G^n} : \sum_{g \in G} A_{g_i, g} = n \}.
\]

For any matrix \(A \in M_n(G)\) we define three vectors \(r(A), c(A), p(A) \in \mathbb{N}^{G^n}\) by

\[
r(A) = \left( \sum_{g \in G} A_{g, g_i, g} \right),
\]

\[
c(A) = \left( \sum_{g \in G} A_{g, g, g_i} \right),
\]

\[
p(A) = \left( \sum_{g \in G} A_{g, g, g_i + g_{g, g_i}} \right).
\]

(5)

To each ordered triple \((x, y, z) \in G^n \times G^n \times G^n\) we associate the matrix

\[
\hat{\psi}(x, y, z) = A_{y, x}^x \in M_n(G)
\]

where

\[
(A_{y, x}^x)_{g_i, g_j} = \left| \{ k : (y - x)k = g_i, (z - x)k = g_j \} \right|.
\]

Note that \(\psi(y - x), \psi(z - x)\) and \(\psi(z - y)\) are uniquely determined by the \(A_{y, x}^x\):

\[
\psi(y - x) = r(A_{y, x}^x), \quad \psi(z - x) = c(A_{y, x}^x), \quad \psi(z - y) = p(A_{y, x}^x).
\]

(6)

If we define

\[
X_A := \{ x, y, z \in G^n : \hat{\psi}(x, y, z) = A \}
\]

for \(A \in M_n(G)\), we have the following.

**Lemma 3.1.** The sets \(X_A, A \in M_n(G)\), are the orbits of \(G^n \times G^n \times G^n\) under the action of \(H\).

**Proof:** Let \(x, y, z \in G^n\) and let \(\hat{\psi}(x, y, z) = A\). For \(h = \pi(\cdot) + v \in H\) we have from (1)

\[
(\hat{\psi}(hx, hy, hz))_{g_i, g_j} = \left| \{ k : (hy - hx)k = g_i, (hz - hx)k = g_j \} \right| = |\{ k : (\pi(y - x))k = g_i \}| = |\{ k : (\pi(z - x))k = g_j \}|
\]

\[
(\pi(z - x))k = g_j = |\{ k : (z - x)k = g_i, (z - x)k = g_j \}|
\]

\[(A_{y, x}^x)_{g_i, g_j} = (\hat{\psi}(x, y, z))_{g_i, g_j},\]

which implies
\[ \hat{\psi}(x, y, z) = \hat{\psi}(hx, hy, hz) \]

for any \( h \in H \).

Let \( A \in \text{M}_n(G) \). To show that \( H \) acts transitively on \( X_A \) it suffices to show that for every \((x, y, z) \in X_A\) there is \( h \in H \) such that \((hx, hy, hz)\) only depends on \( A \). For convenience, we denote \( \varphi(y - x) = \alpha \) and \( \varphi(z - x) = \beta \). Let \( \pi_0 \in S_n \) be such that

\[
\pi_0(y - x) = u_0 = 0 \quad \ldots \quad 0 
\]

Thus, \( h = \pi_1 \pi_0(\bullet) - \pi_1 \pi_0(x) \).

Denote the stabilizer of \( 0 \in G^n \) in \( H \) by \( H_0 \). For \( A \in \text{M}_n(G) \), let \( M_A \) be the \(|G|^n \times |G|^n\) matrix defined by:

\[
(M_A)_{yz} = \begin{cases} 
1, & \text{if } \hat{\psi}(0, y, z) = A, \\
0, & \text{otherwise.}
\end{cases}
\]

Note that

\[
M_A^T = M_A^r.
\]

Let \( \tau_{G^n} \) be the set of matrices

\[
\sum_{A \in \text{M}_n(G)} x_A M_A,
\]

where \( x_A \in \mathbb{C} \). From the Lemma 3.1 it follows that \( \tau_{G^n} \) is the set of matrices that are stable under permutations \( \sigma \in H_0 \) of the rows and columns, i.e., for any \( \sigma \in H_0 \) and \( M_A \),
\[(M_\lambda)_{y,z} = (M_\lambda)_{yz, az} \cdot \]

Hence \(\tau_{G^n}\) is a complex matrix algebra called the \textit{centralizer algebra} of \(H_0\). Since

\[M_\lambda M_\beta = 0 \text{ if } c(A)\neq r(B) .\]

it follows that \(\tau_{G^n}\) is a noncommutative algebra. The \(M_\lambda\) constitute a basis for \(\tau_{G^n}\), and hence

\[\dim \tau_{G^n} = |M_n(G)| = \binom{n+|G|^2-1}{|G|^2-1}. \]

Note that the algebra \(\tau_{G^n}\) contains the Bose-Mesner algebra \(A_{G^n}\); for \(\gamma \in N_n^G\) we have (recall (5))

\[D_\gamma = \sum_{\lambda \in M_n(G), \gamma(\lambda) = \gamma} M_\lambda, \]

Let \(\tau\) denote the Terwilliger algebra of the association scheme \((G^n, R)\) (with respect to 0). It is the complex matrix algebra generated by the adjacency matrices of the scheme \(\{D_\gamma\}, \in N_n^G\) and the diagonal matrices \(\{E_\gamma^*\}, \in N_n^G\) defined by

\[(E_\gamma^*)_{xx} = \begin{cases} 1 & \text{if } (0, x) \in R_\gamma \\ 0 & \text{otherwise} \end{cases} \]

\textbf{Theorem 3.2.} The algebras \(\tau_{G^n}\) and \(\tau\) coincide.

\textbf{Proof:} We have already seen in (7) that \(\tau_{G^n}\) contains the adjacency matrices \(D_\gamma\).

Note that

\[E_\gamma^* = M_{\Delta_{\gamma}}, \]

where \(\Delta_{\gamma} = \text{diag}(\gamma_0, \gamma_{g^2}, \ldots, \gamma_{g^{|G|}}) \in M_n(G)\). Hence \(\tau\) is a subalgebra of \(\tau_{G^n}\). Now we show the reverse inclusion. For \(\gamma \in N_n^G\) with \(\gamma_{g^k} \geq k\) and \(g \in G, g_i \neq g_j\) define

\[\gamma(k, g_i, g_j) \in N_n^G \text{ by} \]

\[(\gamma(k, g_i, g_j))_{g^l} = \begin{cases} \gamma_{g^l} & \text{if } l \neq i, j, \\ \gamma_{g^i} - k & \text{if } l = i, \\ \gamma_{g^j} + k & \text{if } l = j. \end{cases} \]
Also define the zero-one matrices:

\[ N_y(k,g_i,g_j) = E^*_r D_{(n-k,0,\ldots,0,k,0,\ldots,0)} E^*_r(k,\bar{g}_i \bar{g}_j) \]

where at the index of the matrix \( D \), \( k \) appears in the \( (g_j - g_i) \) coordinate. Observe that

\[ (N_y(k,g_i,g_j))_{y,z} = 1 \leftrightarrow (0, y, z) \in X_\lambda, \]

Where

\[ (A)_{g_i,g_j} = \begin{cases} 
\gamma_{g_l} & \text{if } l = m \text{ and } l \neq i, \\
\gamma_{g_l-k} & \text{if } l = m = i, \\
k & \text{if } (l, m) = (i, j), \\
0 & \text{otherwise}.
\end{cases} \]

I. Semidefinite Programming Bound

For \( h \in H \) denote the characteristic vector of \( h(C) \) by \( X^{(hc)} \) (taken as a column vector). For a word \( x \in G^n \), let \( h_x \in H \) be any automorphism with \( h_x(x) = 0 \), and define

\[ R_x = \frac{1}{|H_0|} \sum_{\sigma \in H_0} X^\sigma(h_x(C)) (X^\sigma(h_x(C)))^T. \]

Next define the matrices \( R \) and \( R' \) by

\[ R : = \frac{1}{|C|} \sum_{x \in C} R_x, \]

\[ R' : = \frac{1}{|G|^n - |C|} \sum_{x \in G^n \setminus C} R_x. \]

As the \( R_x \), and hence also \( R \) and \( R' \), are convex combinations of positive semidefinite matrices, they are positive semidefinite. By construction, the matrices \( R_x \), and hence the matrices \( R \) and \( R' \) are invariant under permutations \( \sigma \in H_0 \) of rows and columns and hence they are elements of the algebra \( T_0^\ast \). Define the numbers

\[ \lambda_\lambda := |(C \times C \times C) \cap X_\lambda| \]

and let

\[ \mu_a := |\{(0) \times G^n \times G^n \} \cap X_\lambda| \]

be the number of nonzero entries of \( M_a \). It is easy to see that

\[ \mu_A = \left( r(A)_{g_1}, r(A)_{g_2}, \ldots, r(A)_{g_{|C|}} \right) \times \prod_{g_i \in G} \left( r(A)_{g_i}, \ldots, (A)_{g_i \times g_{i+1}} \right) \]
Theorem 4.1.

\[ R = \sum_{A \in M_n(G)} x_A M_A, \]

\[ R' = \frac{|C|}{|C|^{n} - |C|} \sum_{A \in M_n(G)} (x_{\Delta_p(A)} - x_A) M_A, \]

where

\[ x_A = (|C|_{\mu_A})^{-1} \lambda_A. \]

Proof: Denote by \((A, B) := tr(A^*B)\), the standard inner product on the space of complex \( |G^n| \times |G^n| \) matrices. Observe that the matrices \( M_A \) are pairwise orthogonal and that \((M_A, M_B) = \mu_A\) for \( A \in G_M(G) \). Hence

\[ \langle R, M_A \rangle = \frac{1}{|C|} \sum_{x \in C} \langle R_x, M_A \rangle = \frac{1}{|C|} \sum_{x \in C} |\{x\} \times C \times C \cap X_A| = \frac{1}{|C|} \lambda_A \]

implies that

\[ \sum_{(0, y, x) \in X_A} (R)_{y,x} = \lambda_A \]

which is the total number of I's (with repetitions) in positions where \( R \) and \( M_A \) are both nonzero. From the symmetry, each entry in \( M_A \) is counted the same number of times which is \((\mu_A)^{-1}\lambda_A\). Thus the first claim follows:

\[ R = \sum_{A \in M_n(G)} \frac{1}{\mu_A} \langle R, M_A \rangle = \sum_{A \in M_n(G)} x_A M_A. \]

Now, the matrix

\[ T := |C|R + (|G|^{n} - |C|)R' = \sum_{x \in G^n} R_x = \]

\[ = \frac{1}{|H_0|} \sum_{x \in G^n} \sum_{\sigma \in H_0} x^{\sigma(h_x(c))}(x^{\sigma(h_x(c))})^T = \]

\[ = \frac{1}{|H_0|} \sum_{x \in G^n} \sum_{\sigma \in H_0} x^{\sigma(c)}(x^{\sigma(c)})^T = \frac{1}{|H_0|} \sum_{h \in H} x^{h(c)}(x^{h(c)})^T \]
is invariant under permutation of the rows and columns by permutations \( h \in H \) and hence is an element of the Bose-Mesner algebra say

\[
T = \sum_{\psi \in \mathcal{N}_H} b_{\psi} D_{\psi}.
\]

Note that for any \( z \in G^n \) with \( \psi(z) = \gamma \), we have

\[
b_{\psi} = (T)_{h,z} = |C|(R)_{h,z} + (|G|^n - |C|)(R')_{h,z}.
\]

From the definition of \( R' \)

\[
R' := \frac{1}{(|G|^n - |C|)} \sum_{x \in G^n \setminus C} R_x
\]

follows that for \( x \in G^n \setminus C \), holds \( 0 \neq h_x(C) \) and \( 0 \neq \sigma(h_x(C)) \) for any \( \sigma \in H_0 \). Therefore, \( (R')_{h,z} = 0 \) and we obtain

\[
b_{\gamma} = (T)_{h,z} = |C|(R)_{h,z} = |C| \sum_{A \in M_n(G)} x_A(M_A)_{0,x} = |C|x(\gamma),
\]

where \( (\gamma) \) denotes a matrix whose first row is a vector \( \gamma \) and the rest of the rows are zero vectors. Hence we have

\[
(|G|^n - |C|)R' = (T) - |C| R = |C| \sum_{A \in M_n(G)} (x(\mu(A)) - x_A)M_A.
\]

Finally, note that

\[
x(\mu(A)) = (|C|\mu(p(A)))^{-1} \lambda(\mu(A)) = \frac{1}{|C|\mu(p(A))} |C \times C \cap R_p(A)| = \frac{a_{p(A)}}{\nu_p(A)} = (|C|\mu_\Delta(p(A)) = x\Delta p(A).
\]

Since for \( \psi \in \mathcal{N}_H \),

\[
\lambda_{\Delta_\psi} := \left| (C \times C \times C) \cap X_{\Delta_\psi} \right| = \sum_{c \in C} \sum_{c' \in C} 1.
\]

It follows that

\[
\sum_{\psi \in \mathcal{N}_H} \lambda_{\Delta_\psi} = \sum_{\psi \in \mathcal{N}_H} \sum_{c \in C} \sum_{c' \in C} 1 = \sum_{c \in C} \sum_{\psi \in \mathcal{N}_H} \sum_{c' \in C} 1 = \sum_{c \in C} 1 = |C|^2
\]
and hence we have

\[ |C|^2 = \sum_{\gamma \in N_n^G} \lambda_\gamma = \sum_{\gamma \in N_n^G} \sum_{\gamma' \in N_n^G} \mu_{\gamma} x_{\gamma'}. \]

Therefore

\[ |C| = \sum_{\gamma \in N_n^G} \mu_{\gamma} x_{\gamma'} = \sum_{\gamma \in N_n^G} \left( \prod_{0 \leq g \leq e \leq n} \gamma_{g[\gamma]} \right) x_{\gamma'} = \sum_{\gamma \in N_n^G} v_{\gamma} x_{\gamma'}. \]

Now we are ready to formulate the bound.

**Theorem 4.2.** (SDP bound) For any positive integer \( n \) and set \( S \subseteq N_n^G \) such that \((n, 0, \ldots, 0) \in S \)

\[ A_C(n, S) \leq \max \left\{ \sum_{\gamma \in N_n^G} v_{\gamma} x_{\gamma'} \right\} \]

subject to the constraints

\[ x_{\Delta(n, 0, \ldots, 0)} = 1, \]

\[ x_A = 0 \text{ if } \{ r(A), c(A), p(A) \} \notin S, \]

\[ x_A = x_A^R, \]

\[ R \succ 0, R^t \succ 0. \]

**REFERENCES**


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НОВЫЕ ВЕРХНИЕ ГРАНИЦЫ НЕБИНАРНОГО КОДА

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